Prediction of Free Surface Water Plume as a Barrier to Sea-skimming Aerodynamic Missiles: Underwater Explosion Bubble Dynamics

My-Ha D.[1], Lim K.M.[1], Khoo B.C.[1], Willcox K.[2]

[1] National University of Singapore, [2] Massachusetts Institute of Technology

Outlines

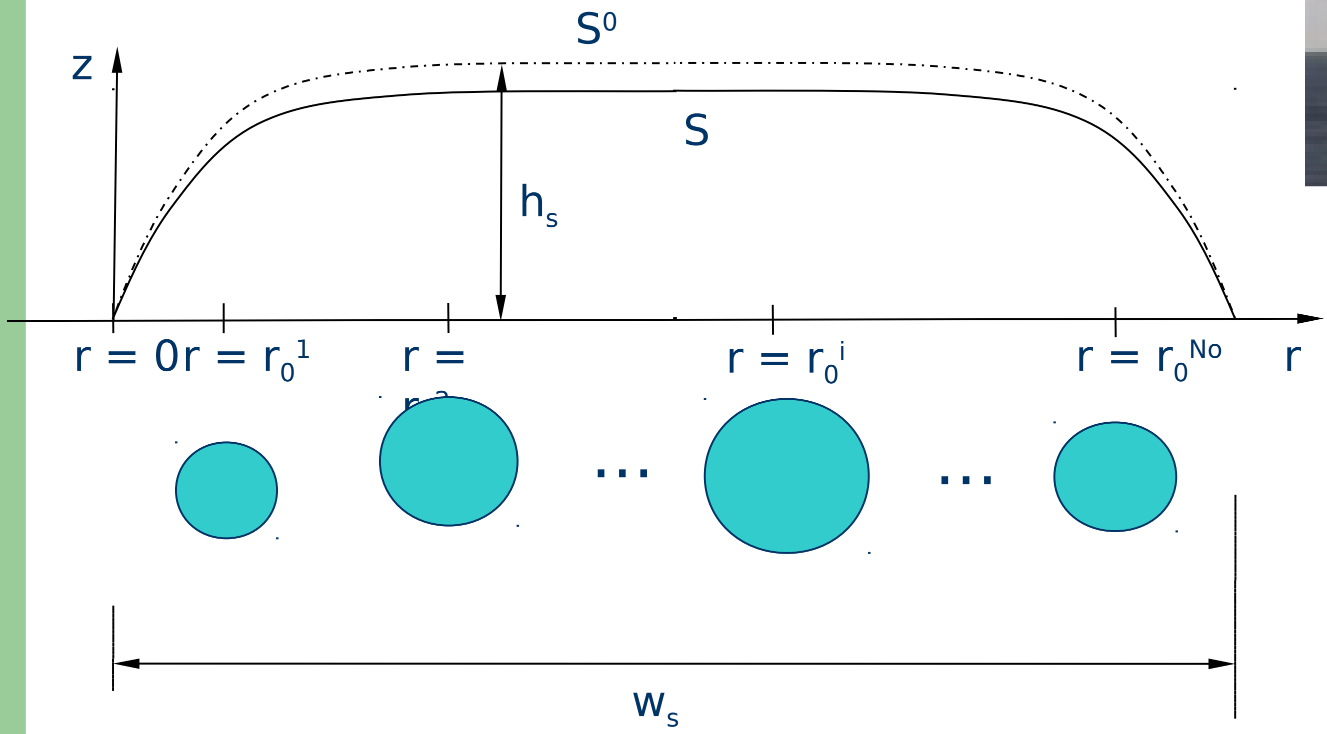
- Problem of water barrier formation
- Simulation of bubble and free surface interaction
- Proper Orthogonal Decomposition (POD)
- Water barrier simulation
- Numerical results
- Conclusions

Problem of water barrier formation

Motivation

- A set of bubbles are created under the water surface
- The free surface is pushed up due to the evolution of the bubbles to create a water plume
- The resultant water plume will act as an effective barrier. It interferes with flying object skimming just above the free surface

Water barrier by underwater explosions



Simulation of bubble and free surface interaction

Mathematical formulation

z

n

Σ

FREE SURFACE

UNPERTURBED FREE SURFACE

(0, 0)

r

Ω

FLUID

BUBBLE

S

n

- Bubble consists of vapor of surrounding fluid and non-condensing gas
- Non-condensing gas is assumed ideal
- Fluid in the domain Ω is inviscid, incompressible and irrotational
- Potential flow satisfies Laplace equation

$$\nabla^2 \Phi = 0$$

Simulation of bubble and free surface interaction

Mathematical formulation

- Integral representation

$$c(x)\Phi(x) = \int_{\partial\Omega} \left(\frac{\partial \Phi(y)}{\partial \mathbf{n}} G(y,x) - \Phi(y) \frac{\partial G(y,x)}{\partial \mathbf{n}} \right) dS$$

Green's function: $$G(y,x) = \frac{1}{|x - y|}$$

- Axisymmetrical formulation as in Wang et al.

$$\mathbf{G} \frac{\partial \Phi}{\partial \mathbf{n}} = \mathbf{H} \Phi$$

Simulation of bubble and free surface interaction

Governing equations

- Kinematic and dynamic boundary conditions

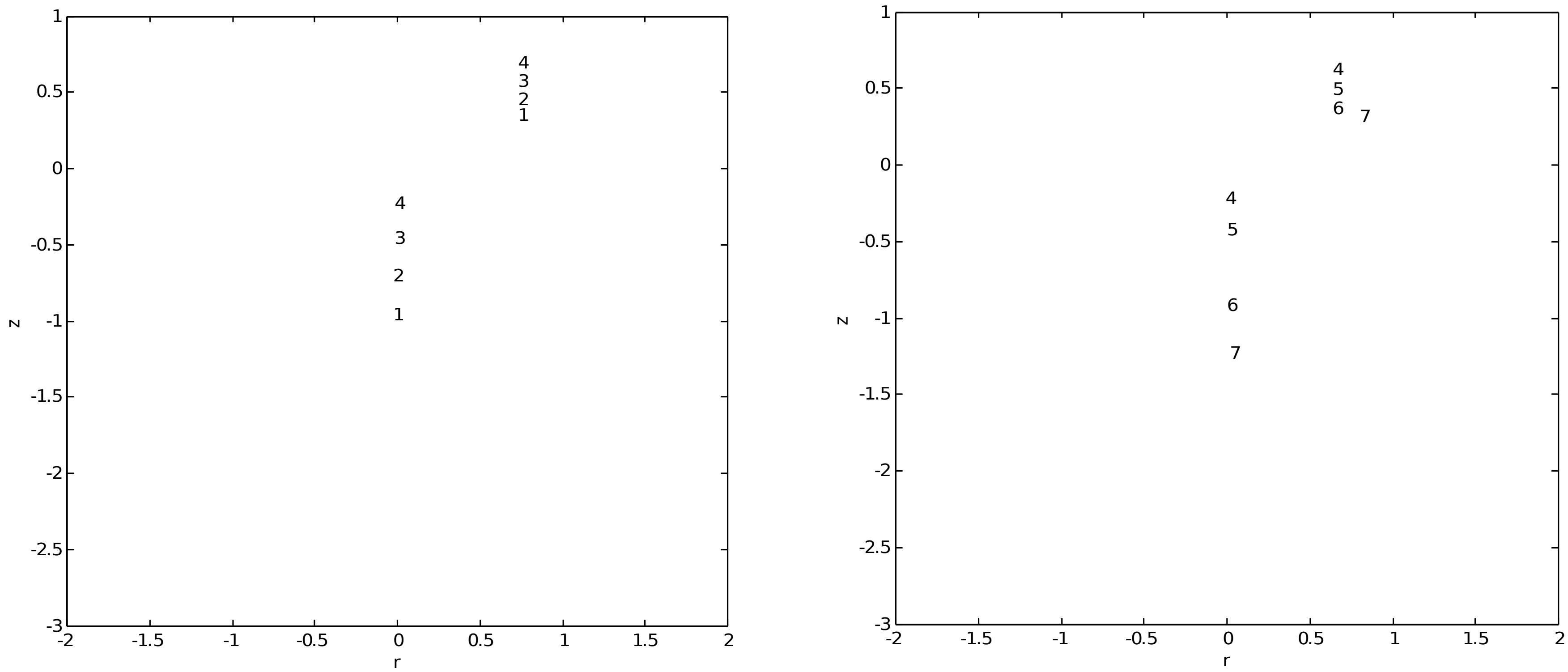
$$\frac{dx}{dt} = \nabla\Phi$$

$$\frac{d\Phi}{dt} = \tfrac{1}{2}\left|\nabla\Phi\right|^2 + \delta^2(z - \gamma) - \varepsilon\left(\frac{V_0}{V}\right)^{\lambda} + 1 \qquad \partial\Omega \in S$$

$$\frac{d\Phi}{dt} = \tfrac{1}{2}\left|\nabla\Phi\right|^2 + \delta^2 z \qquad \partial\Omega \in \Sigma$$

- Solving these equations gives the position and the velocity potential of the nodes on the boundary

Bubble and free surface interaction

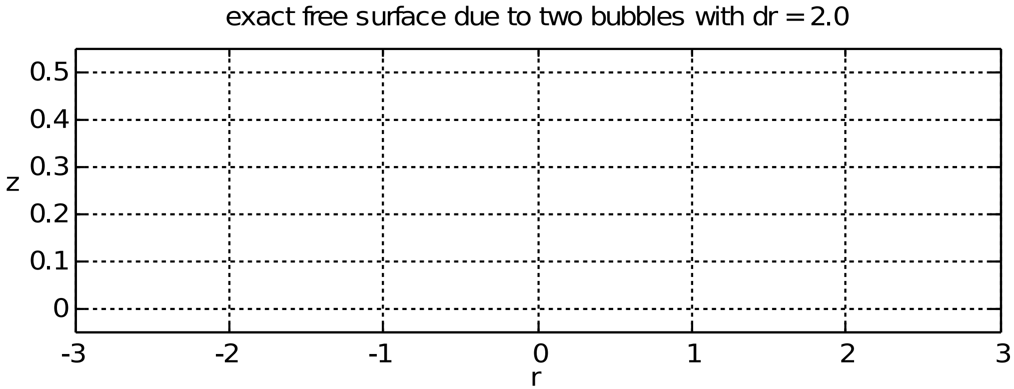


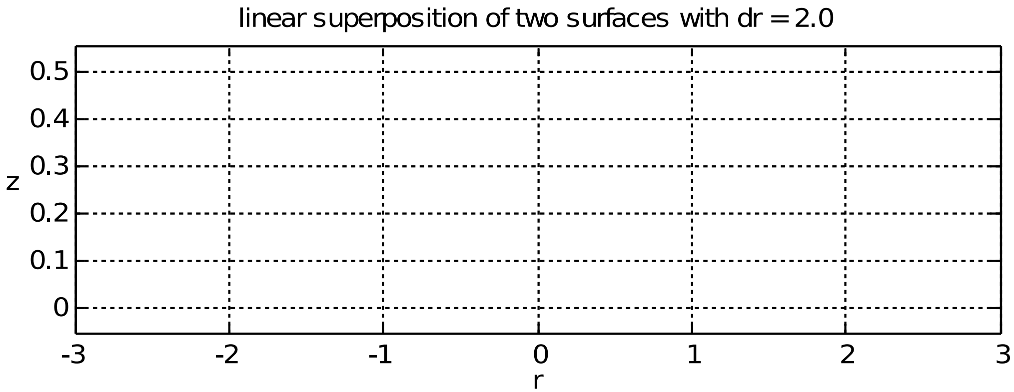
Evolution of bubble and free surface profile for

$$\gamma = -1.25, \epsilon = 100, \delta = 0.3$$

Simulation of bubble and free surface interaction

Linear superposition





Linear superposition of free surfaces formed by two bubbles at distance

$$dr = 2.0$$

$$\gamma = \frac{h}{R_m} = -1.25$$

$$\varepsilon = \frac{P_{b,0}}{P_{ref}} = 100$$

$$\delta = \sqrt{\frac{\rho g R_m}{P_{ref}}} = 0.3$$

Proper Orthogonal Decomposition (POD)

- POD is also known as
  - Principle Component Analysis (PCA)
  - Singular Value Decomposition (SVD)
  - Karhunen-Loève Decomposition (KLD)

- POD has been applied to a wide range of disciplines such as image processing, signal analysis, process identification and oceanography

Proper Orthogonal Decomposition (POD)

Basic POD formulation

- Given a set of snapshots which are solutions of the system at different instants in time
- The basis $\left\{ \varphi_j(x) \right\}_{j=1}^{\infty}$ are chosen to minimize the truncation error due to the construction of the snapshots using M basis functions

$$\max_{\psi} \frac{\left\langle \left| (u,\psi) \right|^2 \right\rangle}{(\psi,\psi)} = \frac{\left\langle \left| (u,\varphi) \right|^2 \right\rangle}{(\varphi,\varphi)}$$

- An approximation is given by

$$u(x) \approx \sum_{j=1}^{q} a_j \varphi_j(x) \qquad q \le M \ \& \ q << N$$

- Given a number of modes, POD basis is optimal for constructing a solution

Method of snapshots

- The POD basis vectors can be calculated as

$$\varphi_j(x) = \sum_{i=1}^{M} b_i^j u_i(x) \qquad j = 1, \dots, M$$

- The vectors b^j satisfies the modified eigenproblem

$$\mathbf{R}b = \lambda b \quad \text{where} \quad \mathbf{R}_{ij} = \frac{1}{M}(u_i, u_j)$$

- Size of **R** is M×M where M is the number of snapshots
  - M is much smaller than N
  - N x N eigenproblem is reduced to M x M problem

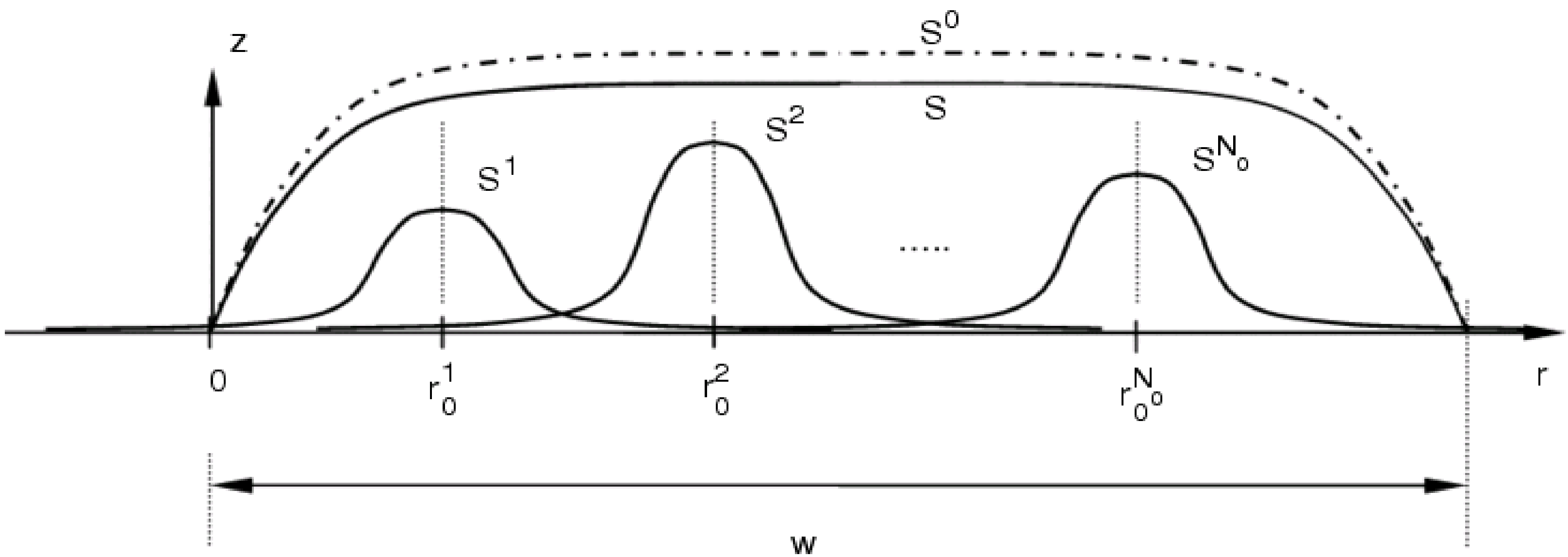
Parametric POD

- Snapshots u^{η_i} are taken corresponding to the parameter value η_i $i=1,\dots,M$
- Basic POD is applied on the set of snapshots $\left\{u^{\eta_i}\right\}_{i=1}^{M}$ to obtain the orthonormal basis $\left\{\varphi_i\right\}_{i=1}^{M}$
- POD coefficient $a_j^{\eta_i} = (\varphi_j, u^{\eta_i})$
- POD coefficient a_j^{η} for intermediate value η not in the sample obtained by interpolation of $a_j^{\eta_i}$
- The prediction of u^{η} corresponding to the parameter value η is given by

$$u^{\eta} \approx \sum_{j=1}^{q} a_j^{\eta} \varphi_j \qquad u(\varepsilon,\gamma,x) \approx \sum_{j=1}^{q} a_j(\varepsilon,\gamma)\varphi_j(x)$$

Water barrier simulation

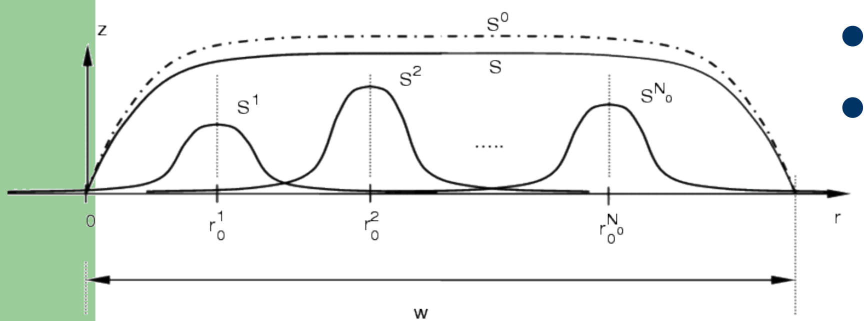
Geometric feature



The problem is considered as an optimization problem that minimizes the difference between constructed surface *S* and desired one S^0

Water barrier simulation

Optimization formulation



- N_0 bubbles are created
- Free surface *S* is approximated by linear superposition of S^k,

k = 1,…,N

- Free surface S^k corresponding to the bubble located at lateral position r_0^k, depth γ^k with strength ε^k is calculated using parametric POD
- Lateral distance between two bubbles is $dr \geq 2.0$
- Strength and depth of the bubbles must be in the ranges of interest

Offline Phase

Collecting snapshots

- Run the described simulation code to collect snapshots
- The snapshot is the free surface shape at the time of maximum height
- The ensemble contains 312 snapshots corresponding to 39 values of initial depth in the range [-1.25, -5.05] with interval step of 0.1, and 8 values of strength in the range [100, 800] with interval step of 100

Online Phase

Optimization formulation

$$
\begin{aligned}
\text{BP :} \quad & \min_{\substack{r_0^k,\, \epsilon^k,\, \gamma^k \\ k\,=\,1,\,\ldots,\,N_0}} && \sum_{i=1}^{M_0} \left[(S - S^0)_i \right]^2 & \\
& \text{s.t.} && S = \sum_{k=1}^{N_0} (\bar{\phi}^k + \sum_{j=1}^{q} a_j^k \phi_j^k) & \\
& && \bar{\phi}^k = \bar{\phi}(r_0^k) & k = 1, \ldots, N_0 \\
& && \phi_j^k = \phi_j(r_0^k) & k = 1, \ldots, N_0; j = 1, \ldots, q \\
& && a_j^k = a_j(\epsilon^k, \gamma^k) & k = 1, \ldots, N_0; j = 1, \ldots, q \\
& && |r_0^{k_1} - r_0^{k_2}| \geq \mathrm{d}r & k_1 \neq k_2; k_1, k_2 = 1, \ldots, N_0 \\
& && \epsilon_{min} \leq \epsilon^k \leq \epsilon_{max} & k = 1, \ldots, N_0 \\
& && \gamma_{min} \leq \gamma^k \leq \gamma_{max} & k = 1, \ldots, N_0
\end{aligned}
$$

Online Phase

Two-stage formulation

- BP involves highly nonlinear representations of the mean of snapshots, POD modes and the interpolation function of POD coefficient
- BP is difficult to solve exactly in a reasonable time even with small number of bubbles involved
- The problem is reformulated as two-stage formulation using the POD feature that the approximated surface is dependent on the depth and strength only through the POD coefficient

Stage 1: Determine bubble positions

$$\text{BPS1}: \quad \min_{\substack{r_0^k,\, a_j^k \\ k=1,\ldots,N_0 \\ j=1,\ldots,q}} \quad \sum_{i=1}^{M_0} \left[(S - S^0)_i\right]^2$$

$$\text{s.t.} \quad S = \sum_{k=1}^{N_0} (\bar{\phi}^k + \sum_{j=1}^{q} a_j^k \phi_j^k)$$

$$\bar{\phi}^k = \bar{\phi}(r_0^k) \qquad k = 1, \ldots, N_0$$

$$\phi_j^k = \phi_j(r_0^k) \qquad k = 1, \ldots, N_0; j = 1, \ldots, q$$

$$|r_0^{k_1} - r_0^{k_2}| \geq \mathrm{d}r \qquad k_1 \neq k_2; k_1, k_2 = 1, \ldots, N_0$$

$$a_j^{min} \leq a_j^k \leq a_j^{max} \qquad k = 1, \ldots, N_0; j = 1, \ldots, q$$

Two methods:
(i) Greedy algorithm
(ii) Approximate Function (AF) algorithm

Stage 1: Greedy algorithm

i. Try all possible placement points
ii. Calculate the resultant free surfaces corresponding to each possible placement points
iii. Choose the point that minimizes the cost function
iv. Repeat process for next bubble

Stage 1: Approximate function (AF) algorithm

- The mean and the first POD mode are approximate by exponential functions in the form of

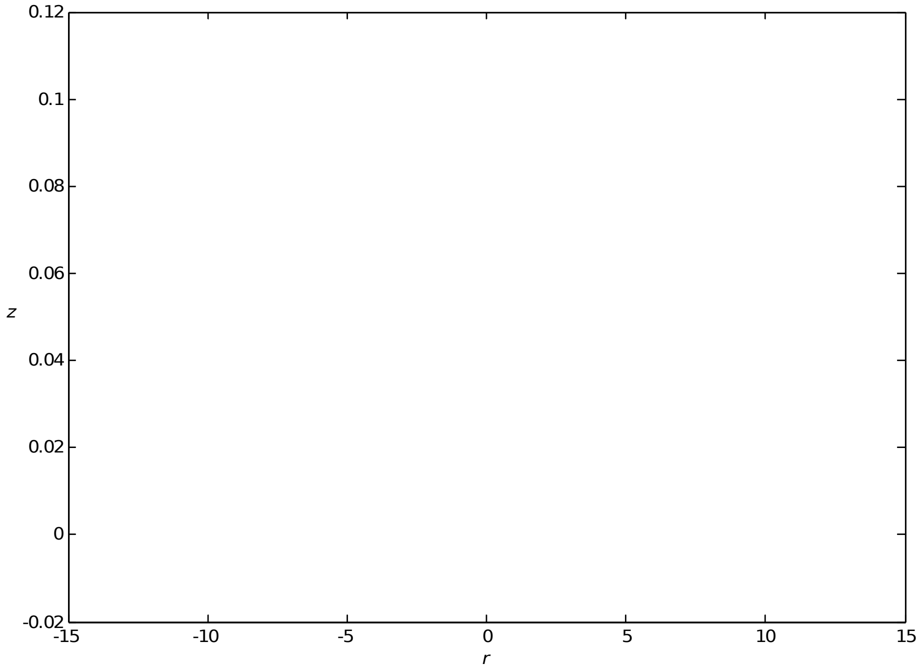
$$f(r) = C_1 e^{-C_2 r^2}$$

- The coefficients C_1, C_2 are obtained by solving the problem

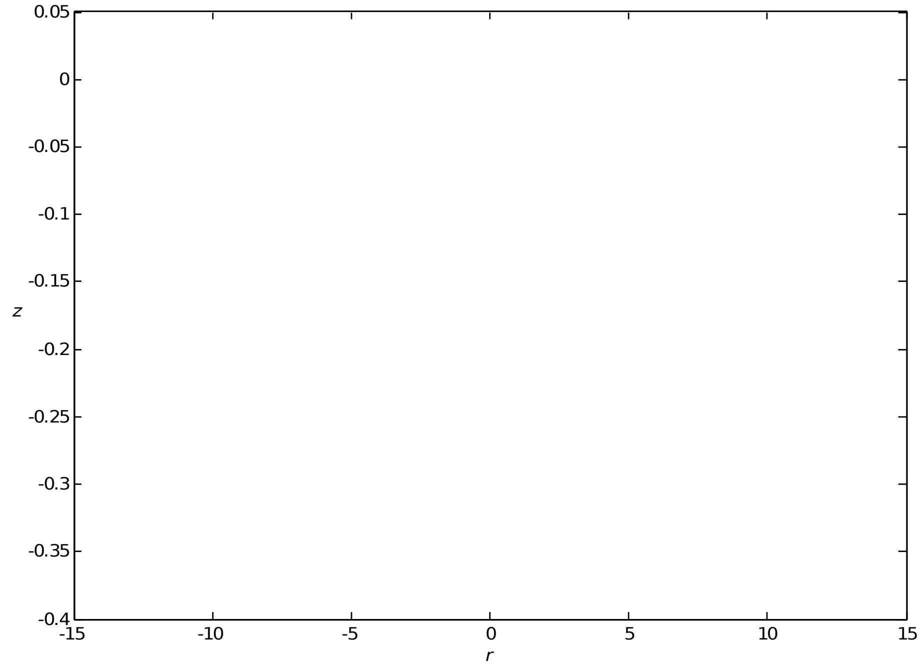
$$\begin{aligned} \mathrm{E1}: \quad & \min_{C_1, C_2} \quad \|(f - f^0)_i\|^2 \\ \text{s.t.} \quad & f(r) = C_1 e^{-C_2 r^2} \end{aligned}$$

Approximate functions

The mean of snapshots



The first POD mode



Approximate exponential functions (blue) compare quite well with exact vectors (red).

Online Phase

Second-stage problem

- Let $\left\{ r_0^k \right\}_{k=1}^{N_0}$ and $\left\{ (a^\star)_j^k \right\}_{k,j=1}^{N_0,q}$ be the solution of Stage 1
- Stage 2 involves minimizing of a piecewise function and this is given by the global minimum of the solutions of its piecewise function elements.

$$\text{Stage 2:} \qquad \min_{\varepsilon^k,\ \gamma^k} \sum_{j=1}^{q} \left[\left(a^*\right)_j^k - a_j \right]^2$$

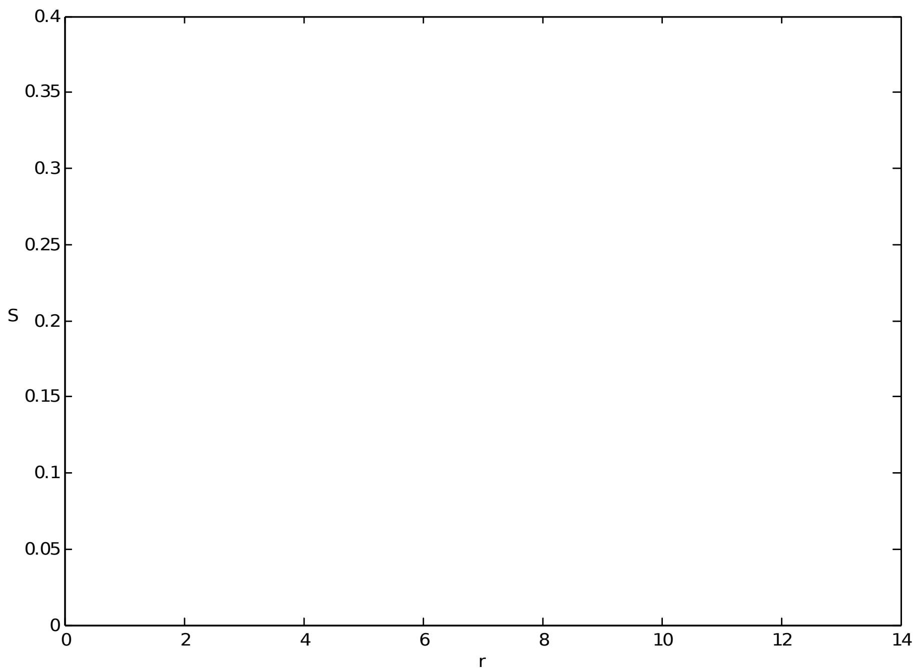
$$\begin{aligned} \text{s.t.} \quad & a_j\left(\varepsilon^k, \gamma^k\right) = a_j^{st}\left(\varepsilon^k, \gamma^k\right) \\ & \varepsilon_t \le \varepsilon^k \le \varepsilon_{t+1} \\ & \gamma_s \le \gamma^k \le \gamma_{s+1} \\ & s = 1, \ldots, n-1;\ t = 1, \ldots, m-1; \\ & j = 1 \ldots q \end{aligned}$$

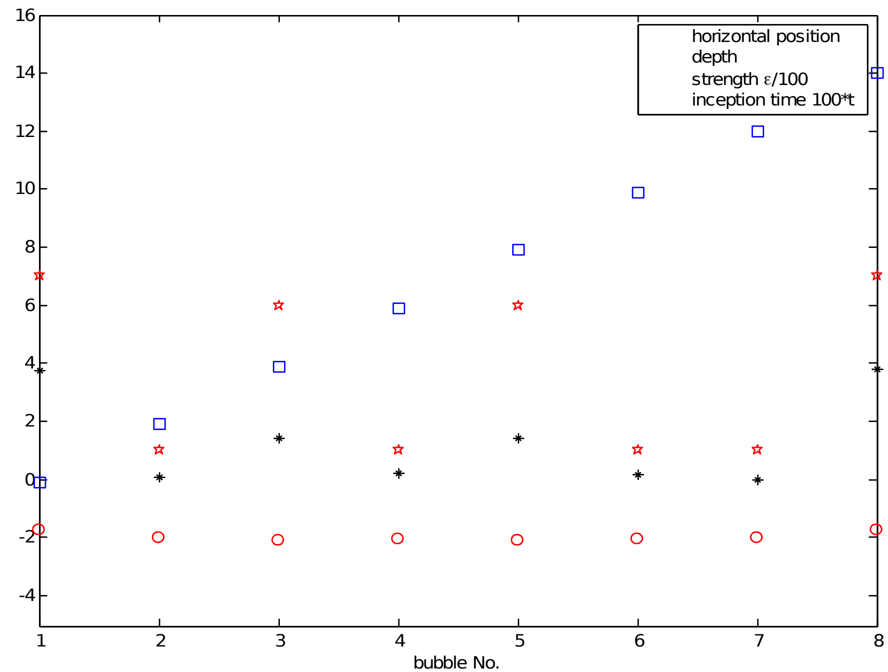
- The bubble problem is fully solved when the lateral position, the depth and the strength of the bubbles are determined.

Numerical results

2D water barrier

Example 1

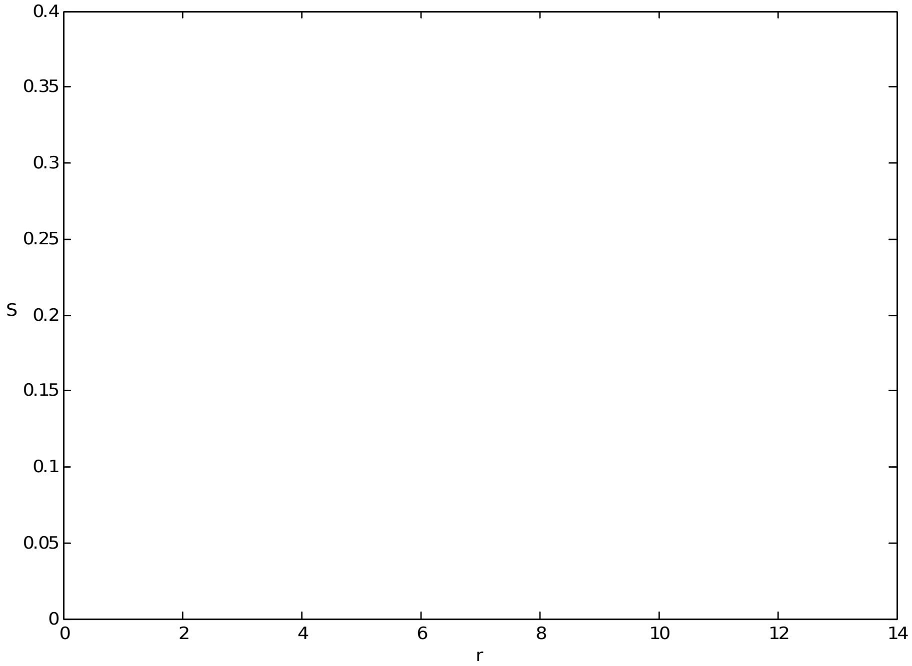


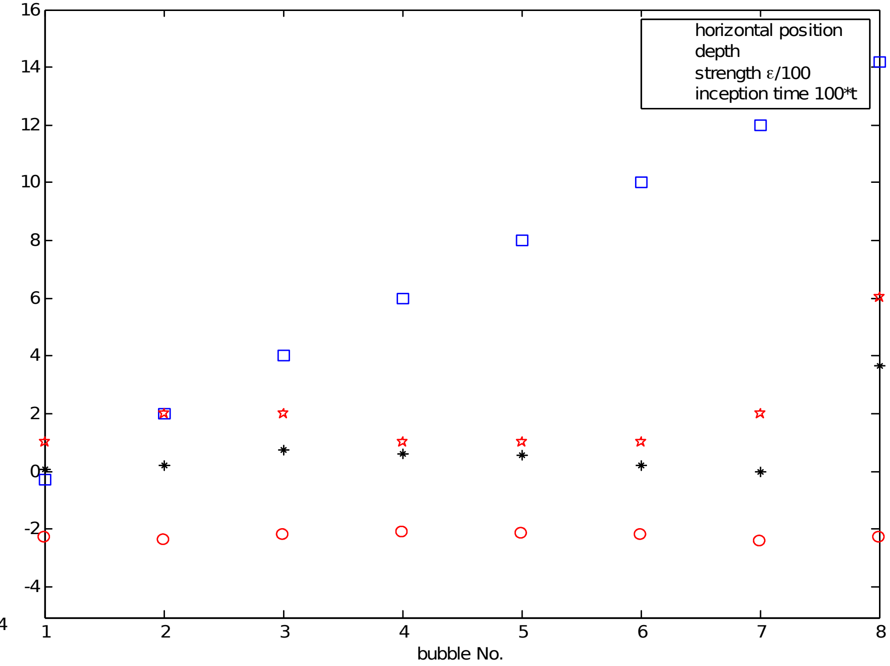


Numerical results

2D water barrier

Example 2

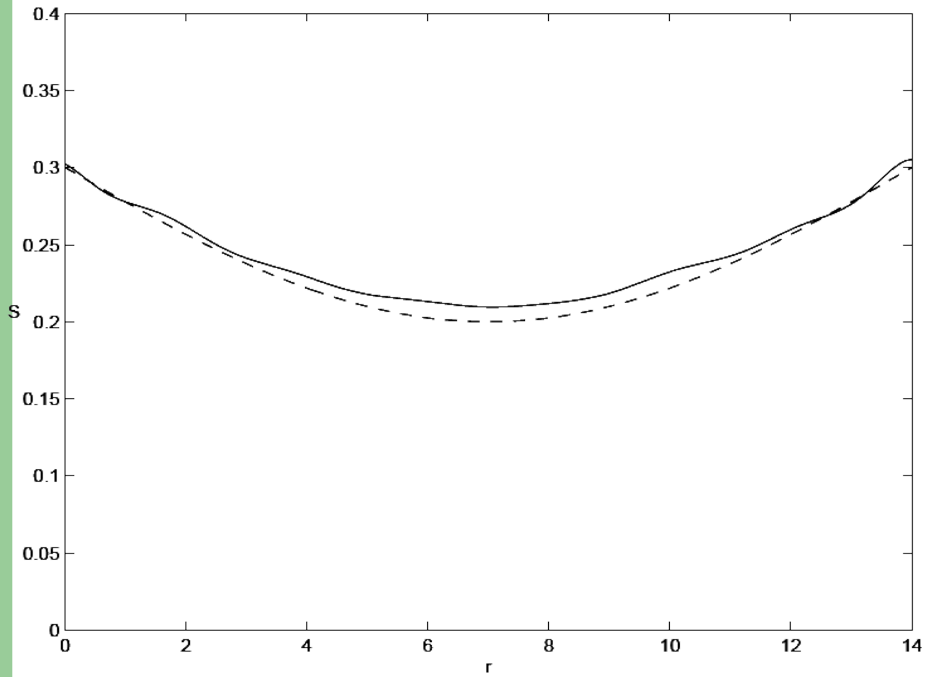


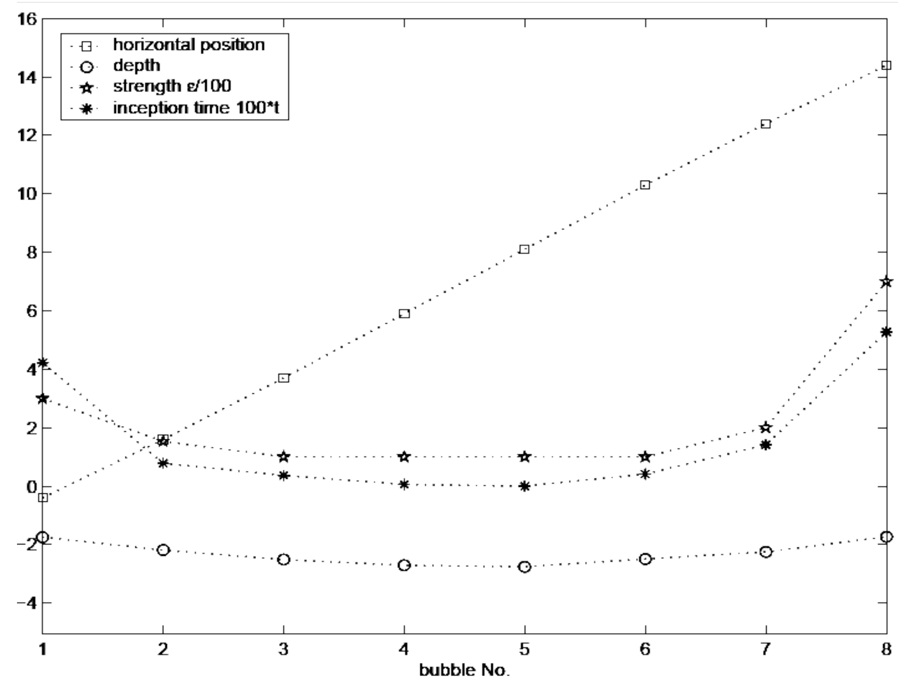


Numerical results

2D water barrier

Example 3

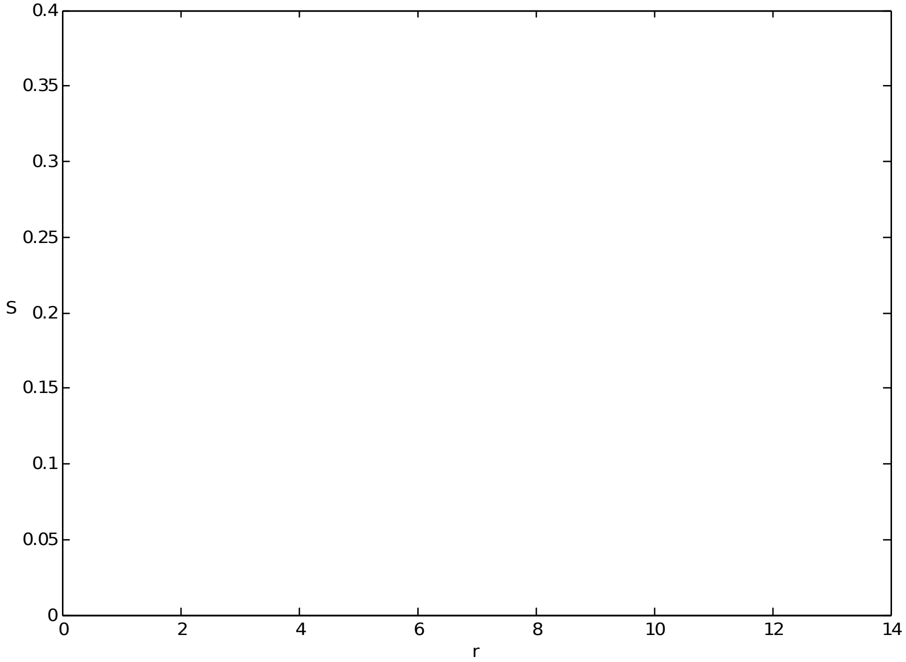


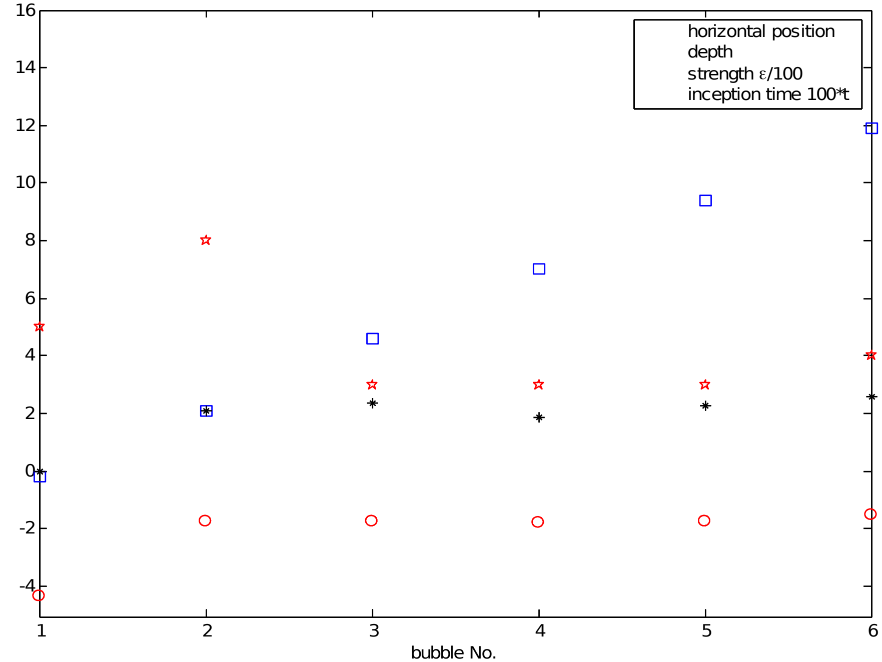


Numerical results

2D water barrier

Example 4





Numerical results

2D water barrier

- Computation time for 2D problems

TABLE I
COMPUTATION TIME FOR 2D PROBLEMS

No. of bubbles	Computation Time
5 bubbles	14.0s
6 bubbles	16.0s
7 bubbles	20.0s
8 bubbles	25.0s
Full simulation of 1 bubble	60.0s

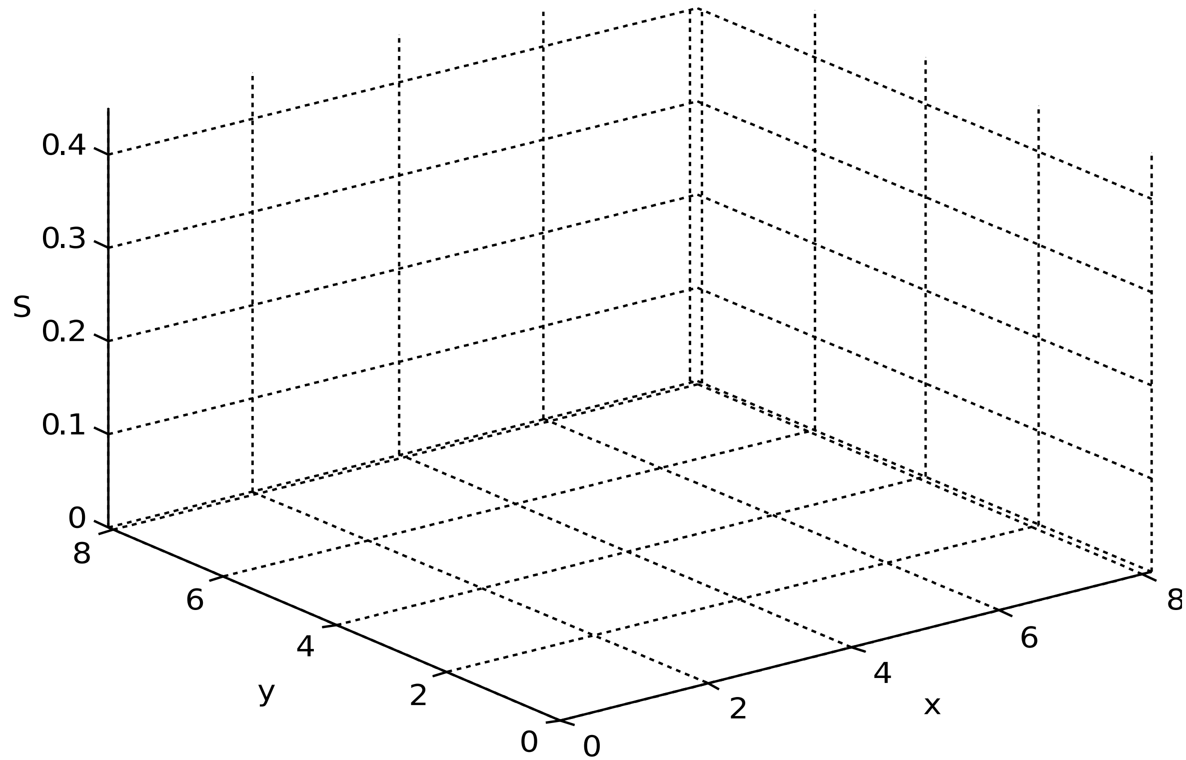
Computation time for solving two-dimensional optimization problem in the domain of [0,14], 141 grid points (grid size 0.1) using AF algorithm. Solution is obtained by running a LOQO program on a Intel Xeon 2.8GHz processor, RAM 2.0Gb

- Computation time increases fairly linearly with no. bubbles

Numerical results

3D water barrier

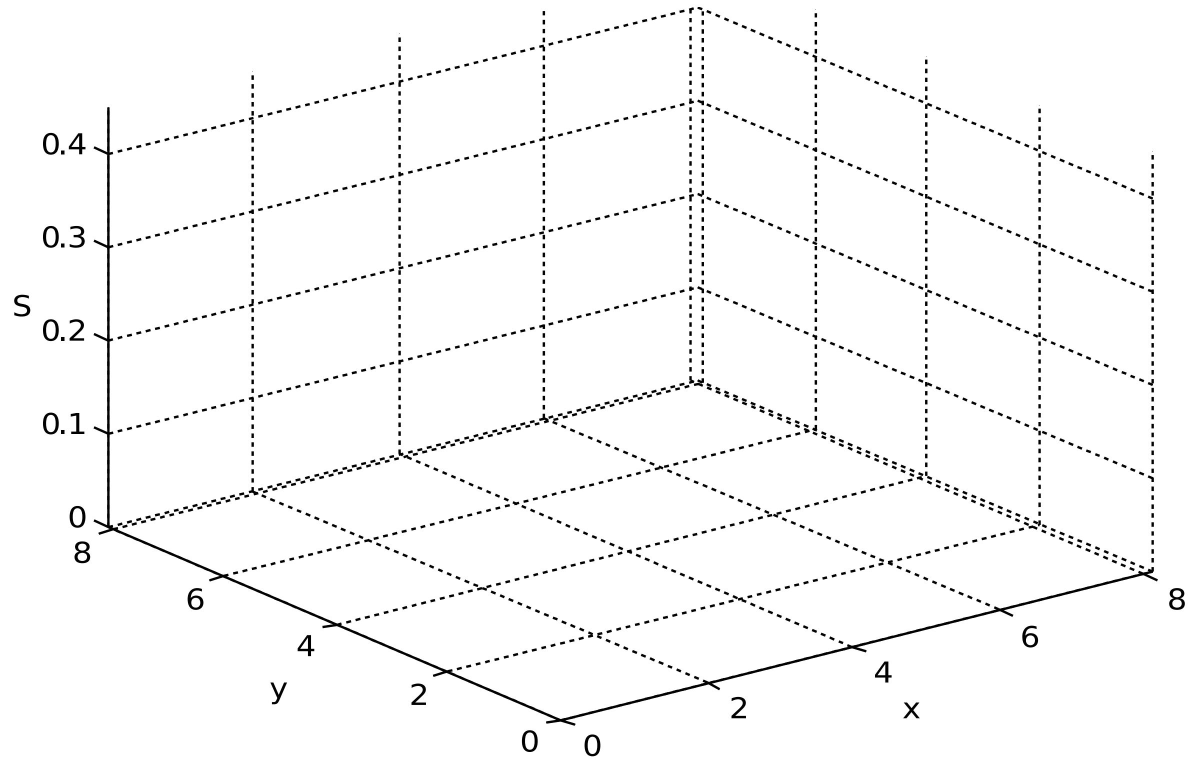
Desired surface



Numerical results

3D water barrier

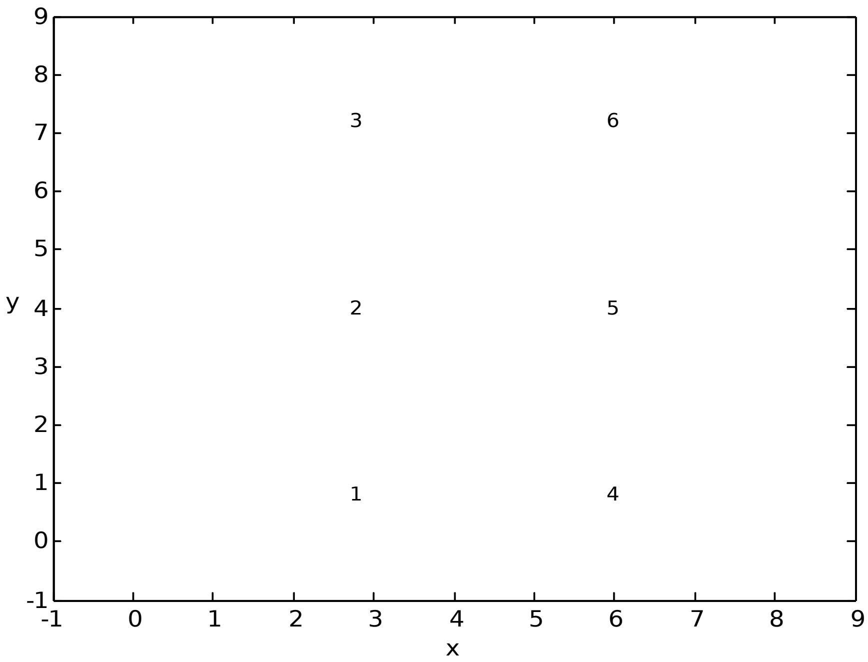
Constructed free surface corresponding to 6 bubbles

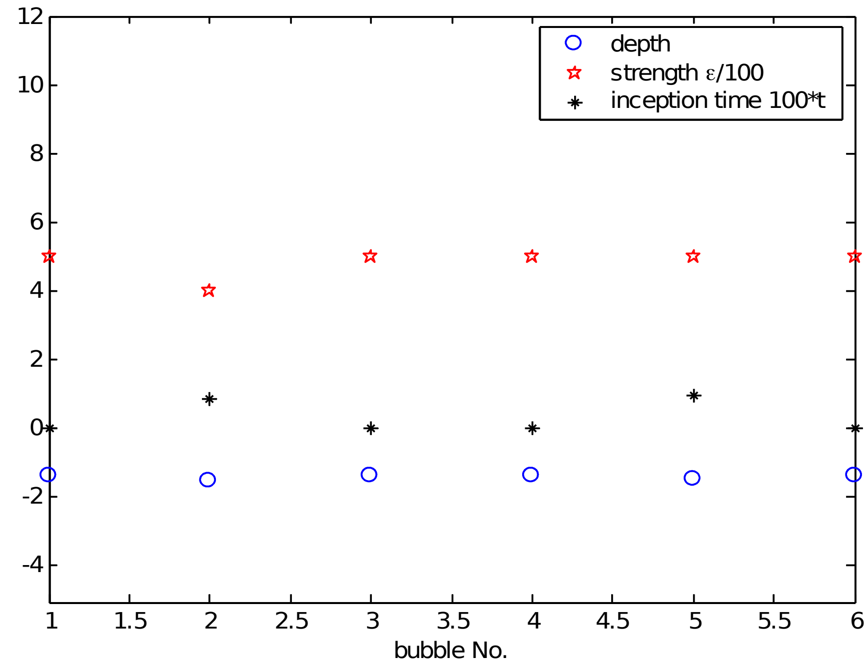


Numerical results

3D water barrier

Parameters for 6 bubbles

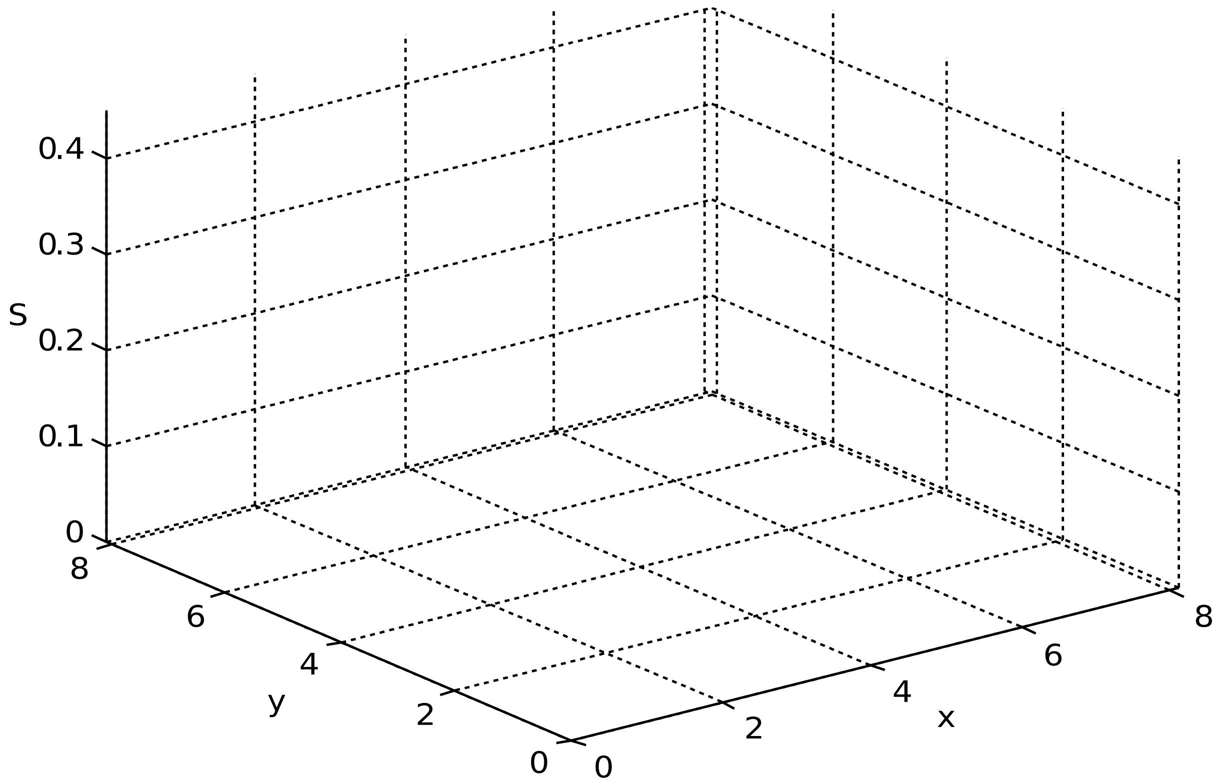




Numerical results

3D water barrier

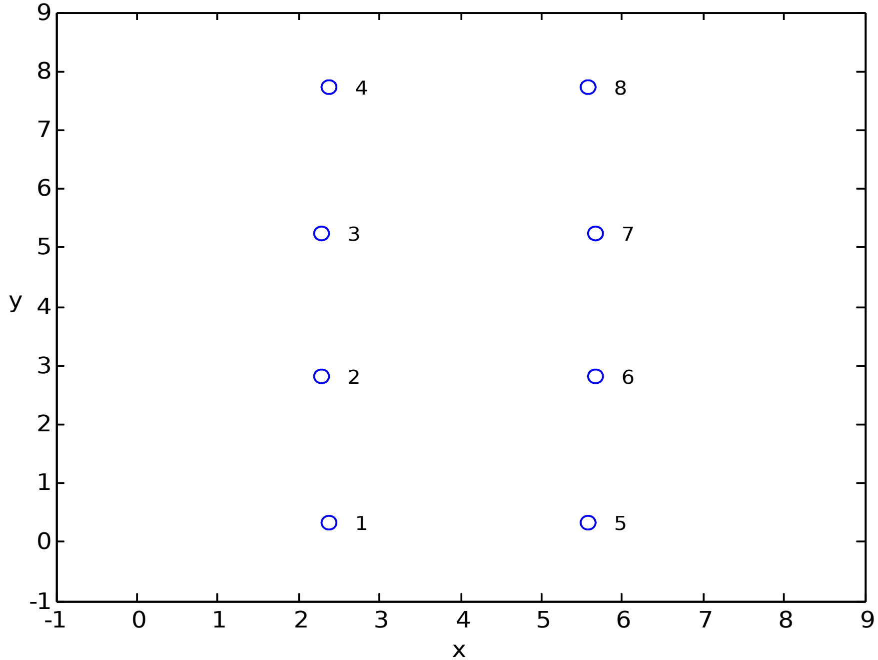
Constructed free surface corresponding to 8 bubbles

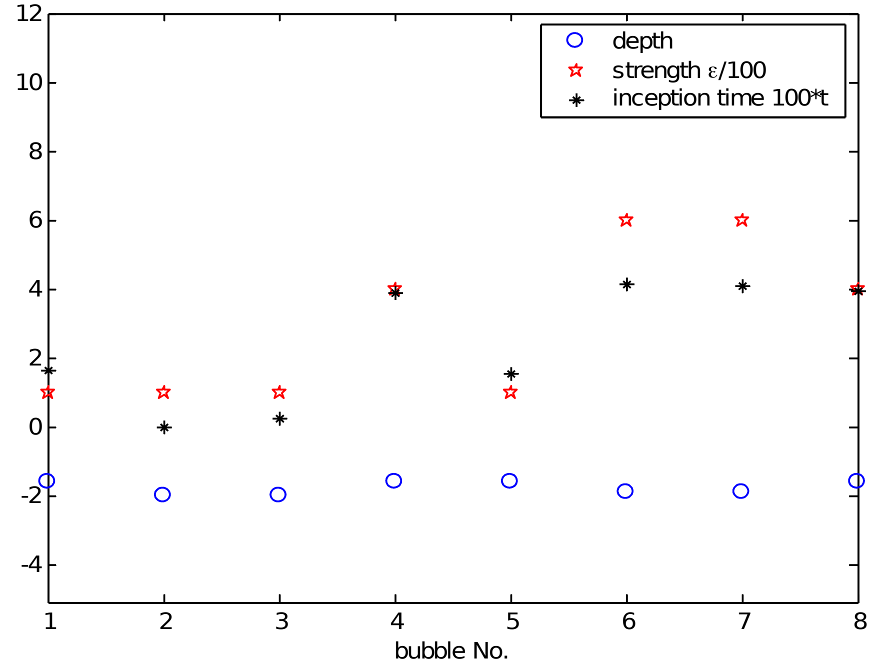


Numerical results

3D water barrier

Parameters for 8 bubbles

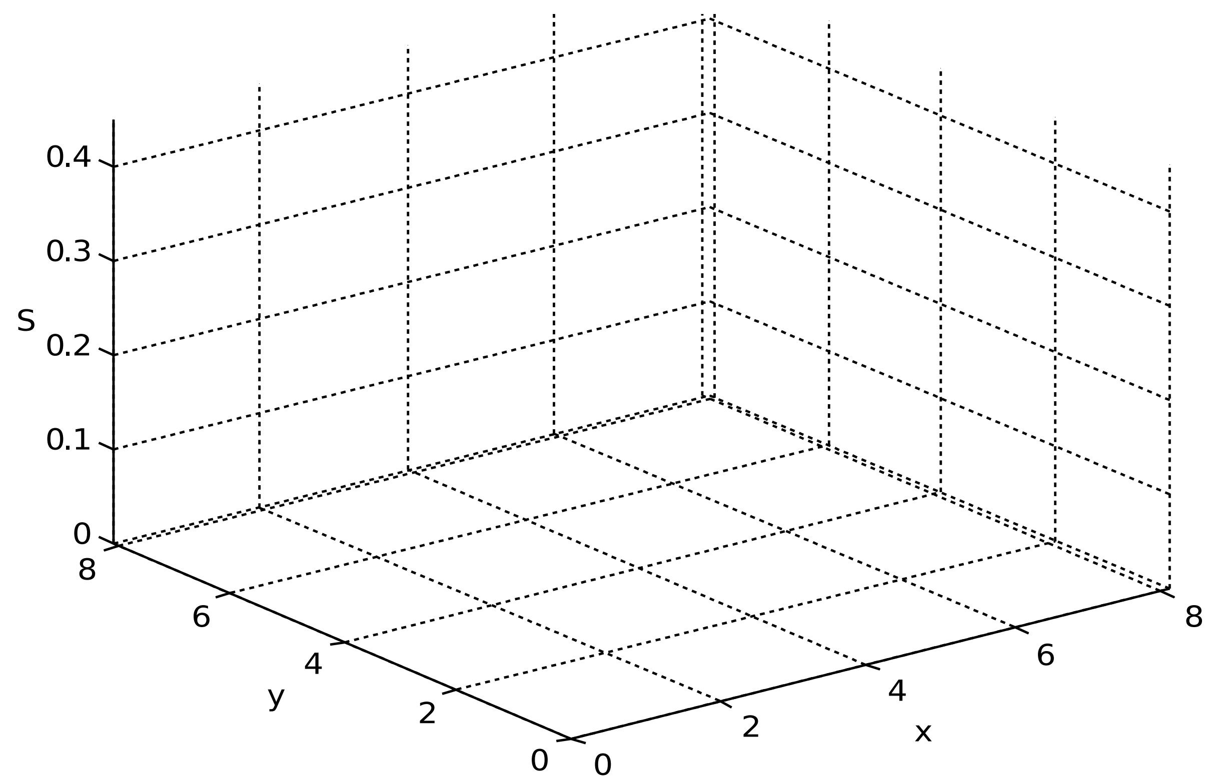




Numerical results

3D water barrier

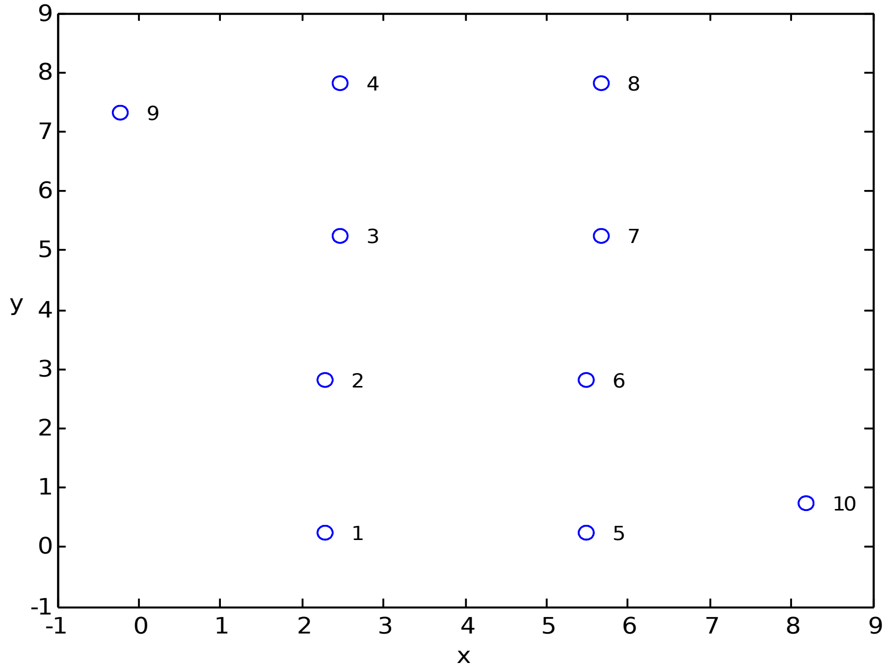
Constructed free surface corresponding to 10 bubbles

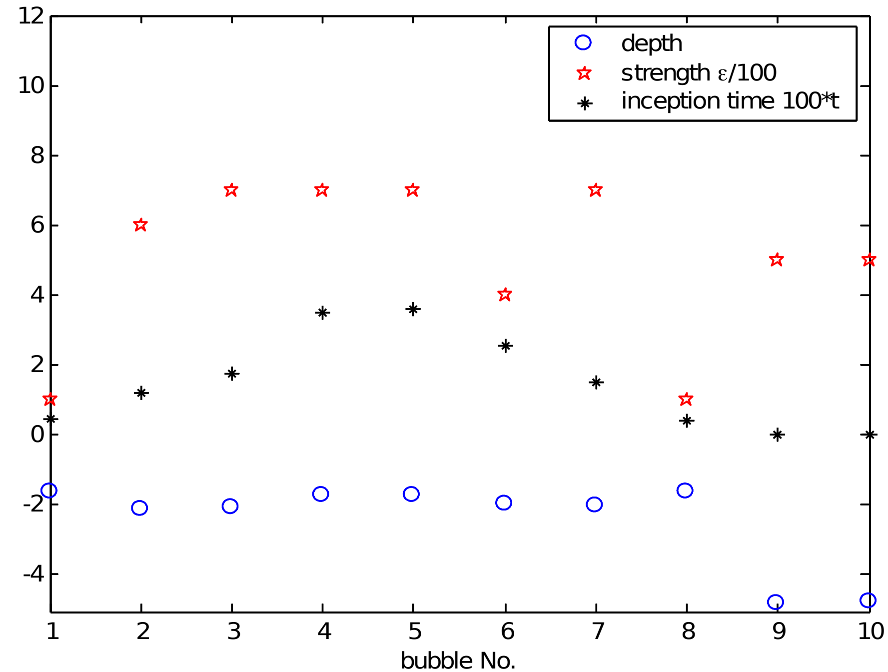


Numerical results

3D water barrier

Parameters for 10 bubbles





Numerical results

3D water barrier

- Computation time for 3D problems

TABLE II
COMPUTATION TIME FOR 3D PROBLEMS

NO. BUBBLES	COMPUTATION TIME
6 bubbles	24.0s
8 bubbles	32.0s
10 bubbles	50.0s
12 bubbles	83.0s
14 bubbles	170.6s
Full simulation of 1 bubble (1717 nodes)	280s

Computation time for solving three-dimensional optimization problems with different numbers of bubbles in a domain of [0 8]x[0 8] grid 17x17 using AF algorithm. Solution is obtained by running a LOQO program on a Intel Xeon 2.8GHz processor, RAM 2.0Gb

- AF algorithm is still efficient for reasonable size problems

Conclusions

- The problem of water barrier formation can be posed as an optimization problem coupled with underwater gas bubbles and free air-water interface
- POD with linear interpolation in parametric space is an excellent tool for constructing a reduced-order model
- Solution algorithm is very efficient for 2D problems and reasonable size 3D problems
- The optimization problem may has many local minima. A robust procedure for finding initial guess may result in a better final solution